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Particle acceleration at ultrarelativistic shocks: an eigenfunction method

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ABSTRACT

We extend the eigenfunction method of computing the power-law spectrum of particles accelerated at a relativistic shock front to apply to shocks of arbitrarily high Lorentz factor. In agreement with the findings of Monte-Carlo simulations, we find the index of the power-law distribution of accelerated particles which undergo isotropic diffusion in angle at an ultrarelativistic, unmagnetized shock is $s = 4.23 \pm 0.01$ (where $s = -\mathrm{d}\ln f/\mathrm{d}\ln p$ with f the Lorentz invariant phase-space density and p the momentum). This corresponds to a synchrotron index for uncooled electrons of $\alpha = 0.62$ (taking cooling into account $\alpha = 1.12$), where $\alpha = -\mathrm{d}\ln F_\nu/\mathrm{d}\ln\nu$, F_ν is the radiation flux and ν the frequency. We also present an approximate analytic expression for the angular distribution of accelerated particles, which displays the effect of particle trapping by the shock: compared with the non-relativistic case the angular distribution is weighted more towards the plane of the shock and away from its normal. We investigate the sensitivity of our results to the transport properties of the particles and the presence of a magnetic field. Shocks in which the parameter σ (the ratio of Poynting to kinetic energy flux) upstream is not small are less compressive and lead to larger values of s.

Subject headings: acceleration of particles—galaxies: jets—gamma rays: bursts—plasmas—pulsars: general—shock waves

1. Introduction

The theory of diffusive acceleration at shock fronts was first developed in 1977 – for reviews see Drury (1983), Jones & Ellison (1991) and Blandford & Eichler (1987) – and was quickly applied to many astrophysical problems. In its simplest form, this theory assumes that accelerated particles diffuse in space upstream and downstream of a discontinuity in the flow velocity of the plasma. This assumption requires the ratio of the plasma speed to the particle speed to be a small quantity. The theory is, therefore, restricted to nonrelativistic flows. Although it was already well-known that relativistic flows exist and contain accelerated particles, it took some time for the theory to be extended into this domain. Until recently, the situation for relativistic shocks was that a semi-analytic eigenfunction method had been developed and was capable of computing the expected power-law index of accelerated particles for shock fronts moving at Lorentz factors Γ of up to roughly 5, assuming various models describing the way in which the particles diffuse in pitch angle in the upstream and downstream plasmas (Kirk & Schneider 1987; Heavens & Drury 1988). This method also provided the full angle and space dependence of the highly anisotropic distribution function. Several sets of Monte-Carlo simulations had also been performed for the special case of isotropic pitch-angle diffusion, and these confirmed the semi-analytic results. For a review see Kirk & Duffy (1999).

Motivated mainly by developments in the field of gamma-ray bursts, Monte-Carlo simulations of acceleration at highly relativistic ($\Gamma > 5$) shocks have recently been presented by Bednarz & Ostrowski (1998) and Gallant et al (1998, 2000), indicating that the index s of the power-law spectrum tends for large Γ to a value close to 4.2, as had been speculated by Heavens & Drury (1988). In order to provide an independent check on the simulation results and to extend them to cover more general diffusion coefficients, we present in this paper a new eigenfunction method, suitable for arbitrary shock speeds. Compared with the original method, the expansion of the distribution function in the new method converges much more rapidly, so that for most purposes only a single eigenfunction is required. This enables rapid computation of s over a wide range of parameter space and for a variety of angular diffusion coefficients. For highly relativistic shocks, the angular distribution at the shock front and at all points upstream is given by a simple analytic expression, in which the details of the scattering and the downstream equation of state enter only through the value of s.

It is at first sight not obvious that the first-order Fermi process will operate at an ultrarelativistic shock front. Unless the system is fine-tuned, such a shock front will be 'superluminal' in which case the Fermi process requires the transport of particles across field lines (Begelman & Kirk 1990; Achterberg & Ball 1994; Michalek & Ostrowski 1998). This aspect of the problem and the way in which we parameterize the cross-field transport is discussed in Sect. 2. The new method is then described in Sect. 3. In Sect. 4 we first present results obtained for isotropic angular diffusion at strong shocks in an ideal gas in which the magnetic field is dynamically unimportant. We then consider the spectrum produced by a decrease in the shock compression due to a finite magnetic field strength. Finally, we investigate the modifications introduced by anisotropic diffusion coefficients. In Sect. 5 we summarize our results and briefly discuss their application to shock fronts in astrophysical systems.

2. Particle transport at ultrarelativistic shocks

In the case of a relativistic parallel shock as considered by Kirk & Schneider (1987), the particle distribution is assumed gyrotropic (i.e., independent of gyro-phase) and particles diffuse in pitch angle. The particles are injected with a Lorentz factor greater than that of the shock, but still small compared to that of the accelerated particles we are interested in. They are treated as test-particles which do not affect the plasma conditions. The stationary transport equation satisfied in both the upstream and downstream regions is then

$$\Gamma(u+\mu)\frac{\partial f(p,\mu,z)}{\partial z} = \frac{\partial}{\partial\mu}\left[D_{\mu\mu}\frac{\partial f(p,\mu,z)}{\partial\mu}\right] \tag{1}$$

where u is the speed of the plasma in units of the speed of light, measured in the shock frame (in which the distribution is assumed stationary), $\Gamma = (1-u^2)^{-1/2}$, $D_{\mu\mu}$ is the pitch-angle diffusion coefficient and $f(p,\mu,z)$ is the (Lorentz invariant) phase-space density, assumed to be a function of the momentum p, the cosine of the pitch angle μ and a single Cartesian coordinate z. Both the momentum and pitch-angle variables are measured in the local rest frame of the plasma, whereas position is measured relative to the shock front in the shock frame.

However, a parallel shock front is a special case which is unlikely to be realized for high Γ. Upon boosting from the rest frame of the upstream plasma to the shock frame, the component of the magnetic field in the plane of the shock is amplified by a factor $\sim\Gamma$, whereas that along the shock normal is conserved. Fine-tuning of the field orientation upstream of the shock is required if it is to remain approximately parallel i.e., sub-luminal (Kirk & Heavens 1989). This, in turn, means that no first order Fermi-type acceleration at all is expected unless there exists a scattering mechanism which enables particles to cross field lines (Begelman & Kirk 1990).

In this paper, we assume such a scattering mechanism exists, and that it can be described by an equation identical to Eq. (1), in which, however, the variable μ no longer describes the (cosine of the) pitch angle of a particle, but instead the cosine of the angle between a particle's direction of motion and the normal to the shock front (which we call the 'direction angle'). Scattering of this kind can be produced by fluctuations in the magnetic field which are of much shorter wavelength than the gyroradius of a particle. The effect of these fluctuations cancels out in the standard treatment of particle transport in a plasma, where one averages over many gyrations (Luhmann 1976). But, in the case of a relativistic shock front, an upstream particle will perform only a fraction of a gyration in between encounters with the front (Gallant & Achterberg 1999), so that the standard treatment does not apply. Downstream, it is widely assumed that a large amplitude turbulent magnetic field is generated by the shock front. If we assume that this field reverses direction on a length scale small compared to the gyro radius, then a section of the trajectory short enough to be considered unperturbed will not resemble a helix, but rather a straight line. In each case it seems reasonable to abandon the pitch-angle description altogether and consider instead diffusion in direction angle.

Equation (1) with isotropic diffusion in direction angle, i.e., $D_{\mu\mu} \propto (1-\mu^2)$ corresponds exactly to the model adopted in the Monte-Carlo simulations of Gallant et al (2000) and approximately to the model used by Bednarz & Ostrowski (1998). Gallant & Achterberg (1999) use this model in the downstream region and, in addition allow for deflection by a uniform magnetic field upstream of the shock.

Isotropic diffusion in direction angle is, however, an idealization, since a preferred direction of the magnetic field – if one exists – is likely to influence the transport. It is widely thought that relativistic shocks will generate a highly tangled field. If this is seeded by the compression of preexisting fluctuations in the upstream medium, then it might be expected that the correlation length of the field in the z direction normal to the shock front is much shorter than that in the x and y directions. Consider the ideal case of a magnetic field $\vec{B} = (B_x(z), B_y(z), B_z)$ which is static in the plasma rest frame and which fluctuates as a function of z only. The statistical properties of interest are specified as follows

$$\left\langle B_x(z')B_x(z+z')\right\rangle = \left\langle B_y(z')B_y(z+z')\right\rangle \equiv S(|z|) \tag{2}$$

with

$$S(z) = 0 \text{ for } z > \ell_{||} \tag{3}$$

The average of the square of the magnetic field $\left\langle {B_\perp}^2\right\rangle$ is defined by

$$\int_{-\infty}^{+\infty} \mathrm{d}z S(|z|) \equiv \left\langle {B_\perp}^2\right\rangle \ell_{||} \tag{4}$$

where $\langle \ldots \rangle$ denotes an ensemble average.

Integrating the equation of motion of a particle of charge e, mass m, velocity v and Lorentz factor γ moving in the x-z plane with direction angle $\arccos\mu$, we find the change in μ after an elapsed time t to be:

$$\Delta\mu = \frac{e}{\gamma mc}\int_0^t \mathrm{d}t' \sqrt{1-\mu^2} B_y(z(t')) \tag{5}$$

In the usual way – see, for example Ichimaru (1973) – we compute the diffusion coefficient by integrating over a time τ long compared to the correlation time of the field (in this case $\ell_{||}/\mu v$) but short enough to allow us to use the unperturbed trajectory: $\mu =$constant, $x = \sqrt{1-\mu^2}vt$, $y = 0$, $z = \mu vt$. Taking an ensemble average, we find

$$\frac{\left\langle \Delta\mu^2\right\rangle}{\tau} = \frac{e^2}{\gamma^2 m^2 c^2}\frac{1-\mu^2}{|\mu| v}\left\langle {B_\perp}^2\right\rangle \ell_{||} \tag{6}$$

Clearly, this expression is unphysical in the neighborhood of $\mu = 0$, because we have assumed an infinite correlation length in the x and y directions. Eliminating this unphysical behavior we write for the diffusion coefficient

$$D_{\mu\mu} = \frac{e^2}{\gamma^2 v m^2 c^2} \frac{1-\mu^2}{\sqrt{\mu^2 + (\ell_{||}/\ell_{\perp})^2}} \langle B_{\perp}{}^2 \rangle \ell_{||} \qquad (7)$$

where $\ell_{||}$ and $\ell_{\perp}$ are related to the correlation lengths parallel and perpendicular to the shock normal, respectively.

The diffusion coefficient in Eq. (7) is plausible in the downstream medium, which is compressed along the shock normal. In the upstream medium this should not be the case. However, especially for highly relativistic shocks, it is immaterial which diffusion coefficient is used there, since only that part of the function $D_{\mu\mu}$ is of importance which lies close to $\mu = -1$. In Sect. 4.3 we investigate the effect of the strongly anisotropic diffusion coefficient given by Eq. (7) on the acceleration process.

3. The eigenfunction method

Consider now an infinite plane shock front, so that in the upstream region $z < 0$ and $u = u_-$ and in the downstream region $z > 0$ and $u = u_+$, with both $u_\pm$ positive and $u_- > u_+$.

Since there is no intrinsic momentum scale in Eq. (1), it follows that only the boundary conditions can introduce one into the solution. If the space boundaries are far from the shock, and the lower momentum boundary (the injection momentum) is well below the range of interest, the intrinsic spectrum of particles accelerated by the shock is a power law $f \propto p^{-s}$. The problem is to find the index s, which depends on the plasma speeds (upstream and downstream) and the function $D_{\mu\mu}$.

Separating the variables p, z and μ in Eq. (1) results in an eigenvalue problem for the angular part of the distribution:

$$\frac{\partial}{\partial\mu}\left[D_{\mu\mu}\frac{\partial}{\partial\mu}Q_i(\mu)\right] = \Lambda_i(u+\mu)Q_i(\mu) \ , \qquad (8)$$

with the boundary conditions that the eigenfunctions be regular at the singular points $\mu = \pm 1$. The general solution for the distribution function is then written

$$f = \sum_{i=-\infty}^{\infty} a_i p^{-s} Q_i(\mu) \exp(\Lambda_i z/\Gamma) \qquad (9)$$

and is valid in each half-space $z \geq 0$ and $z \leq 0$. At the shock front ($z = 0$) Liouville's Theorem and the appropriate Lorentz transformations for the particle momentum and direction relate the upstream and downstream distributions. The eigenfunctions Q_i in Eqs. (8) and (9) fall into two families: those with positive eigenvalue, $\Lambda_i > 0$, which are labeled $i = 1, \ldots \infty$ and those with negative eigenvalue, labeled $i = -1, \ldots - \infty$. In addition, there is a special eigenfunction $Q =$

constant, $\Lambda = 0$ for which we choose $i = 0$. The eigenfunctions obey an orthogonality relation

$$\int_{-1}^{1} (u+\mu) Q_i Q_j d\mu = 0 \qquad \forall i \neq j \ . \tag{10}$$

Imposing the condition that the distribution function far downstream (i.e. as $z \to \infty$) does not diverge, the distribution function for $z > 0$, simplifies to

$$f^{+} = \sum_{i=-\infty}^{0} a_i^{+} \, (p_{+})^{-s} \, Q_i^{+} \exp(\Lambda_i^{+} z/\Gamma^{+}) \ . \tag{11}$$

where the super- (or sub-)script "+" denotes a downstream quantity. Far upstream ($z \to -\infty$) the distribution function should not only be regular, but there should also be no incoming accelerated particles. Thus we can write

$$f^{-} = \sum_{j=1}^{\infty} a_j^{-} \, (p_{-})^{-s} \, Q_i^{-} \exp(\Lambda_j^{-} z/\Gamma^{-}) \ . \tag{12}$$

where the super- (or sub-)script "−" denotes an upstream quantity.

In the original method (Kirk & Schneider 1987), the downstream distribution function f^{+} was expanded up to the term $i = -N$ in Eq. (11). Then, the condition that the distribution vanish for $z \to -\infty$ was satisfied approximately by using the orthogonality relation (10) to project f^{-} onto the eigenfunctions Q_i^{-} and demanding that the terms $i = -N, \ldots 0$ should vanish. This resulted in a set of $N+1$ linear homogeneous algebraic equations for the a_i^{+}. The vanishing of the determinant of this system was used to find s. Thus, only those eigenfunctions were required for which $i \leq 0$, which (except for $i = 0$) are oscillatory in the range in which particles move in the same direction as the flow as seen in the shock frame i.e., $-u_{\pm} < \mu_{\pm} < 1$ These were evaluated using a Galerkin method.

In the new method, we adopt the 'mirror image' of this approach, expanding the upstream distribution function to N terms as an ansatz

$$f^{-} \quad = \quad p_{-}^{-s} \sum_{1}^{i=N} a_i^{-} Q_i^{-}(\mu_{-}) \exp(\Lambda_i^{-} z/\Gamma_{-}), \tag{13}$$

for $z \leq 0$, which fulfills the upstream boundary condition at $z \to -\infty$. We then determine s by projecting onto the functions Q_i^{+} for $i = 1, \ldots N$ and solving the resulting N homogeneous equations. In this case, only those eigenfunctions are required for which $i \geq 1$. Except for $i = 1$ these are oscillatory in the interval $-1 < \mu < -u$. At ultrarelativistic shocks, $u_{-} \to 1$, so that the oscillatory region for upstream eigenfunctions becomes small. Although the eigenfunctions are then difficult to evaluate using Galerkin techniques, the direct numerical integration via shooting method and Prüfer transformation used by Heavens & Drury (1988) works well. Furthermore, an analytic expression for these eigenvalues and eigenfunctions has been given in the limit $u \to 1$ (Kirk

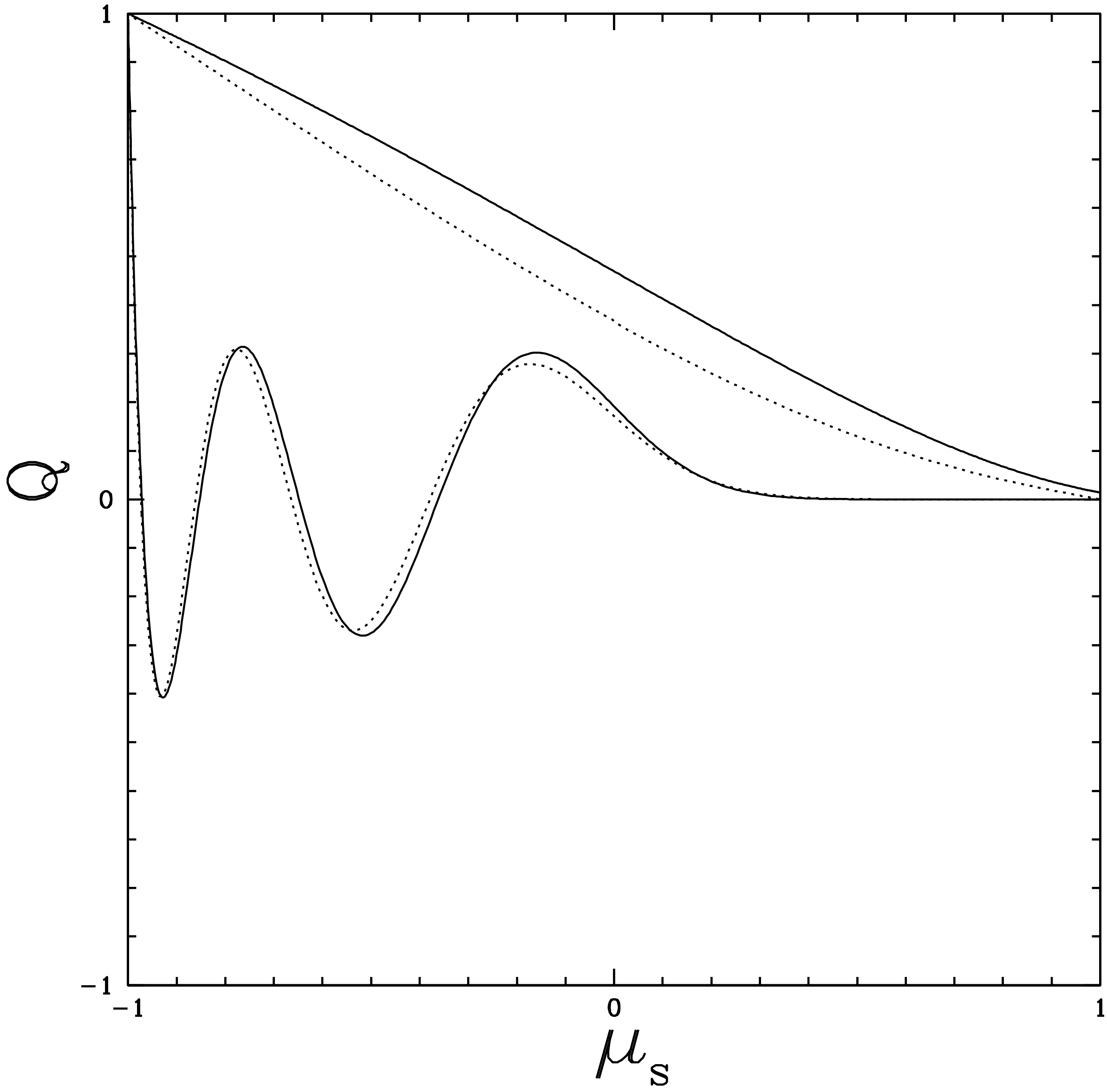

Fig. 1.— The eigenfunctions $i = 1$ and $i = 5$ for isotropic diffusion in direction angle as a function of direction angle measured in the shock frame. The solid line depicts the results of a numerical integration using the Prüfer transformation for a plasma speed $u = 0.5$, the dotted line shows the asymptotic expressions of Eq. (14) for $u \to 1$

& Schneider 1989) and it turns out that the expansion in Eq. (13) converges much more rapidly than that in Eq. (11).

An example of the eigenfunctions for isotropic diffusion in direction angle $D_{\mu\mu} = (1-\mu^2)D(\mu)$, with $D(\mu)$ =constant, is shown in Fig. 1, where the abscissa is chosen to be the (cosine of the) direction angle $\mu_{\rm s}$ measured in the frame which projects $\mu = -u$ onto $\mu_s = 0$, thus stretching the oscillatory range. For $u = u_\pm$, this reference frame corresponds to the shock frame for $Q^\pm$. The i'th eigenfunction possesses $i-1$ roots in the interval $-1 < \mu_{\rm s} < 0$, corresponding to particles streaming against the flow. Only the first eigenfunction $i = 1$ is positive definite. According to Kirk & Schneider (1989), the eigenfunctions for a general diffusion coefficient with $D(\mu)$ are given in the limit $u \to 1$ by the expressions

$$Q_i^\infty(y) \quad = \quad \exp\left[-\epsilon y\sqrt{\Lambda_i^\infty/D(-1)}\right]\sum_{n=0}^{i-1} c_n y^n \ , \tag{14}$$

where $\epsilon = 1-u$ and $y = (1+\mu)/\epsilon$. The asymptotic expression for the eigenvalues is

$$\Lambda_i^\infty = (2i-1)^2 D(-1)/\epsilon^2 \tag{15}$$

and the coefficients are determined by the recursion formula $c_{n+1} = 2(2i-1)(i-n-1)c_n/(n+1)^2$. Figure 1 shows both the numerically calculated eigenfunctions (using the Prüfer transformation and shooting method) for $u = 0.5$ and $D_{\mu\mu} \propto (1-\mu^2)$ and the asymptotic expressions of Eq. (14). For larger values of u, the numerical results for both eigenfunctions and eigenvalues are well approximated by the asymptotic expressions.

Matching the distributions across the shock front i.e., demanding $f^+(p_+,\mu_+,0) = f^-(p_-,\mu_-,0)$ involves the transformations

$$\begin{aligned} p_+ &= \Gamma_{\rm rel}p_-(1+u_{\rm rel}\mu_-) \\ \mu_+ &= (\mu_- + u_{\rm rel})/(1+u_{\rm rel}\mu_-) \end{aligned} \tag{16}$$

where the relative velocity of the upstream medium with respect to the downstream one is

$$u_{\rm rel} \ = \ (u_- - u_+)/(1-u_-u_+)$$

and $\Gamma_{\rm rel} = (1-u_{\rm rel})^{-1/2}$.

The projection of the expansion in Eq. (13) onto the downstream eigenfunctions yields

$$\sum_{j=1}^{j=N} S_{ij}a_j^- \quad = \quad 0 \tag{17}$$

where the matrix S_{ij} is given by

$$S_{ij} \quad = \quad \int_{-1}^{+1} {\rm d}\mu_+(u_+ + \mu_+)(1+u_{\rm rel}\mu_-)^s Q_i^-(\mu_-)Q_j^+(\mu_+) \tag{18}$$

The index s is found from the condition $|S_{ij}| = 0$ and the corresponding distribution function is given by Eq. (13) with those a_i^- which lie in the null-space of S.

It is interesting to note that in the limit $u_- \to 1$, the matrix elements in Eq. (18) may be written

$$S_{ij} \quad \to \quad \int_0^\infty \mathrm{d}y Q_i^\infty(y)(y-1)(y+2)^{-s} Q_j^+(\mu_+) \tag{19}$$

with $\mu_+ = (y-2)/(y+2)$, and $Q^\infty(y)$ given by Eq. (14).

4. Results

4.1. Isotropic diffusion in angle

First of all, we consider a plasma in which the particle transport is described by isotropic diffusion in direction angle ($D_{\mu\mu} = (1-\mu^2)D$ with D constant) and the magnetic field is dynamically unimportant. In this case, the Jüttner/Synge equation of state describes the plasma, and the jump conditions across the shock front must in general be evaluated numerically. However, in the case of a strong shock, a useful analytic expression can be found using an approximate equation of state in which the downstream ratio of specific heats $\hat{\gamma}$ is prescribed (Blandford & McKee 1977):

$$\begin{aligned} w_+/\rho_+ &= \hat{\gamma}(\Gamma_{\rm rel} - 1) + 1 \\ \Gamma_-^2 &= \frac{(w_+/\rho_+)^2(\Gamma_{\rm rel}+1)}{\hat{\gamma}(2-\hat{\gamma})(\Gamma_{\rm rel}-1)+2} \end{aligned} \tag{20}$$

where w is the proper enthalpy density and ρ is the proper (rest-mass) density.

Another straightforward case is that of a relativistic gas both up and downstream, in which case one can derive

$$u_- u_+ = 1/3 \tag{21}$$

This situation is perhaps less likely to occur in practice, since it describes a shock front propagating into a medium in which the rest-mass energy density is negligible, even though the pressure may be important. In the limit $\Gamma_- \to \infty$ both (20) and (21) give $u_+ \to 1/3$, as must any physically acceptable equation of state describing an unmagnetized gas.

Results obtained using these jump conditions are presented in Fig 2. The power-law index s clearly tends to a limiting value as $u_- \to 1$. This limit agrees with the value we find from the asymptotic expression in Eq. (19):

$$s \quad = \quad 4.23 \pm 0.01 \tag{22}$$

where the errors quoted are our rough estimates of the accuracy of the numerical root finding algorithm, and the truncation error arising from the expansion of the distribution.

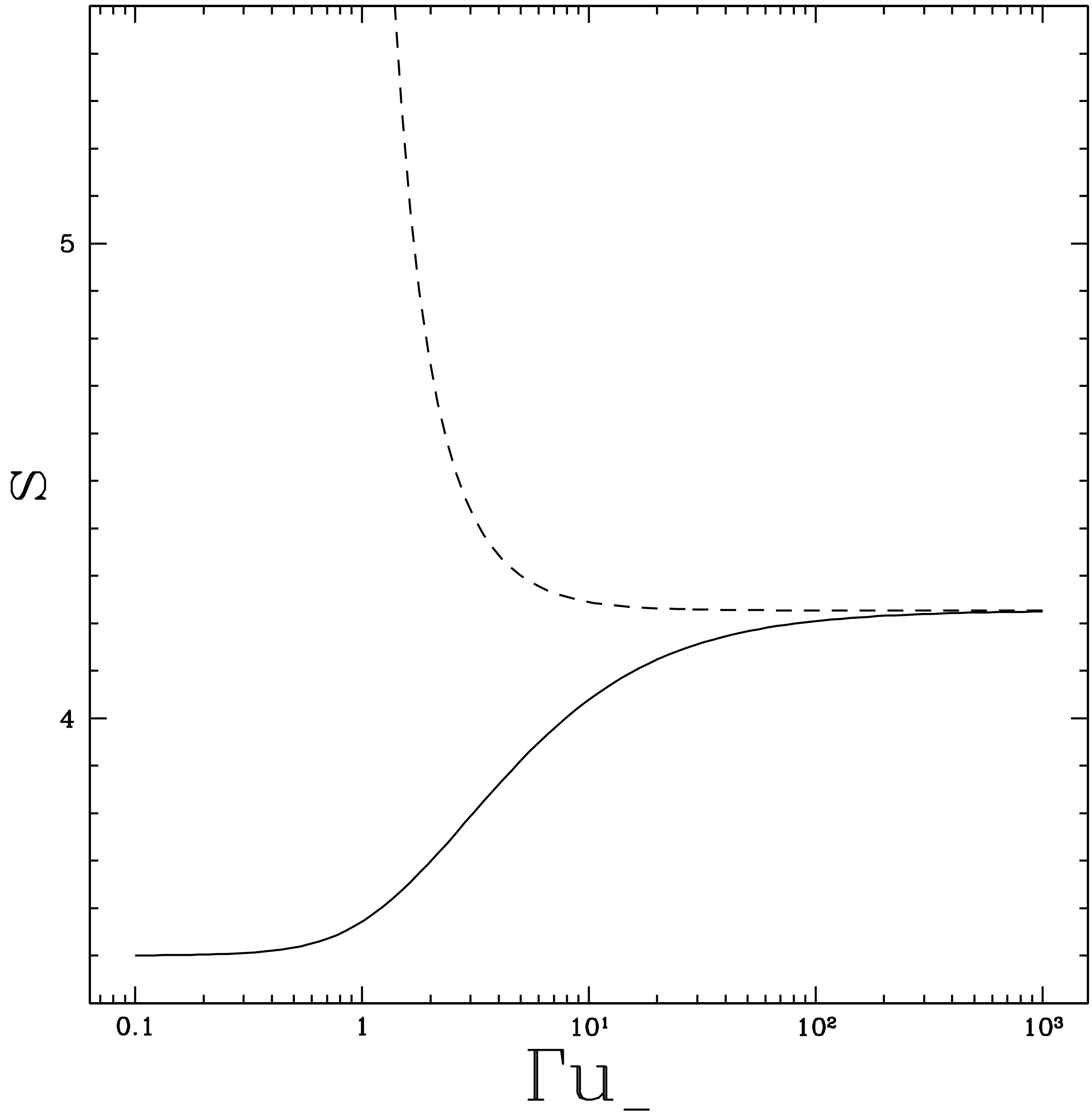

Fig. 2.— The power-law index s as a function of the spatial component of the upstream shock speed $\Gamma_- u_-$ for a strong shock with fixed adiabatic index $\hat{\gamma} = 4/3$ (solid line) and for a shock in a relativistic gas (Eq. 21). The apparent asymptotic value of $s = 4.23$ agrees with explicit computations performed in the limit $u_- \to 1$.

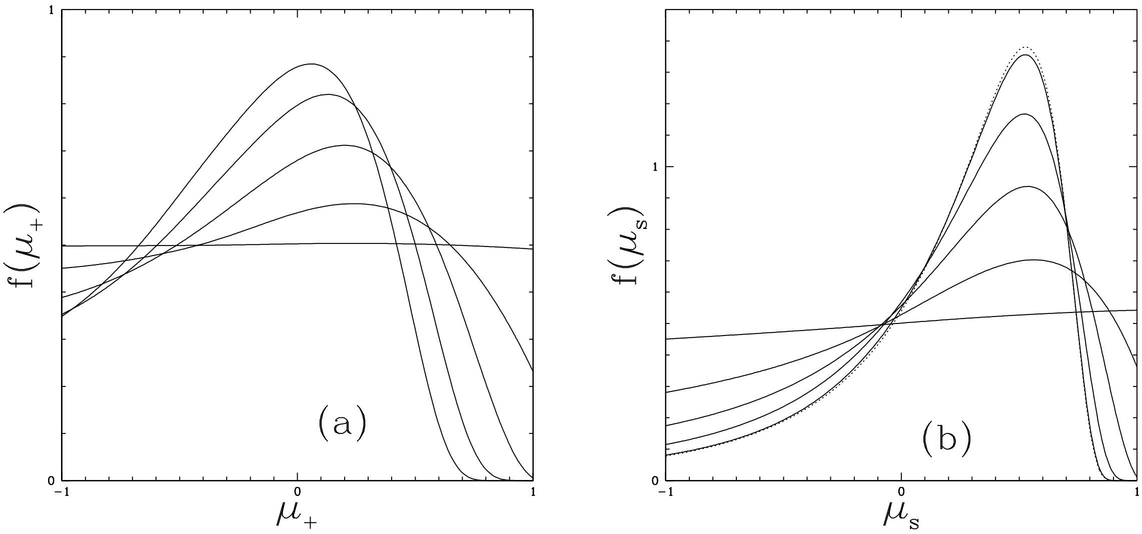

Fig. 3.— The angular distribution of particles at the shock front (a) as seen in the downstream rest frame as a function of the direction angle μ_+ and (b) as seen in the shock rest frame, as a function of the direction angle μ_s. Solid line curves are plotted for five different values of the upstream speed: $\Gamma_- u_- = 0.1$, 0.5, 1, 2, and 10. The dotted line in (b) shows the asymptotic angular distribution given by Eq. (23) for $\Gamma_- u_- \to \infty$ and $s = 4.23$. The almost isotropic distribution at low u_- gives the almost horizontal lines. The peak in the flux of particles is more pronounced at higher $\Gamma_- u_-$. The jump conditions used are those of a strong shock with the Jüttner/Synge equation of state downstream.

The corresponding angular distribution at the shock front is shown in Fig. 3 for several different values of u_-. In the nonrelativistic limit, the distribution function is approximately isotropic and this angular distribution is an almost horizontal line. As the upstream speed increases, a pronounced hole appears in the distribution at the point where particles enter the downstream region along the shock normal. Because upstream particles are caught by the shock before they can be substantially deflected, particles overtaken by a relativistic shock never re-enter the shock front along the normal. To a lesser extent, those particles re-entering the shock from the downstream side are also depleted at more relativistic shocks. These properties express the fact that the return probability of a particle leaving the shock by moving into the downstream region depends on the direction angle. The shock 'captures' preferentially those particles of $\mu_{\rm s} \lesssim 0.6$. The absence of large deflections upstream ensures the energy gain per crossing remains modest after the first shock encounter, at which it may reach a factor Γ_-^2 (Vietri 1995; Gallant & Achterberg 1999).

We checked the convergence properties of our results by changing the number of terms in the expansion. For $N > 5$ we found convergence to within the numerical noise associated with the integration and root finding routines. An accuracy of better than 10% in all cases is already obtained by using just a single term in the expansion, i.e., $N = 1$. Except in the case of anisotropic diffusion (see Sect. 4.3), the angular distribution was also found to be well represented by the first eigenfunction when transformed to the shock frame. For relativistic shocks ($u_- > 0.5$) this distribution can be approximated by the expression given in Eq. (14). When written in the shock frame, the distribution then becomes:

$$f_{\rm s} \quad \propto \quad (1 - \mu_{\rm s} u_-)^{-s} \exp\left(-\frac{1 + \mu_{\rm s}}{1 - u_- \mu_{\rm s}}\right) \tag{23}$$

4.2. Strongly magnetized plasma

The strength of the magnetic field in the upstream or downstream plasma can be conveniently expressed in terms of the Lorentz invariant σ-parameter. This is defined as

$$\sigma \quad = \quad B^2/4\pi w \tag{24}$$

(Kirk & Duffy 1999) where B is the magnetic field and w the enthalpy density, both measured in the local rest frame. In the case of a cold MHD flow carrying a magnetic field perpendicular to the flow direction, this parameter describes the ratio of the Poynting flux to the energy flux density carried by the particles (Kennel & Coroniti 1984a; Michel & Li 1999). A shock moving at speed u_- into the upstream medium with a magnetic field perpendicular to the shock normal then has a fast magnetosonic Mach number $M_{\rm fast}$ (the ratio of u_- to the fast magnetosonic wave speed) given by

$$M_{\rm fast} \quad = \quad u_- \sqrt{\frac{1 + \sigma}{\sigma + v_{\rm s}^2}} \tag{25}$$

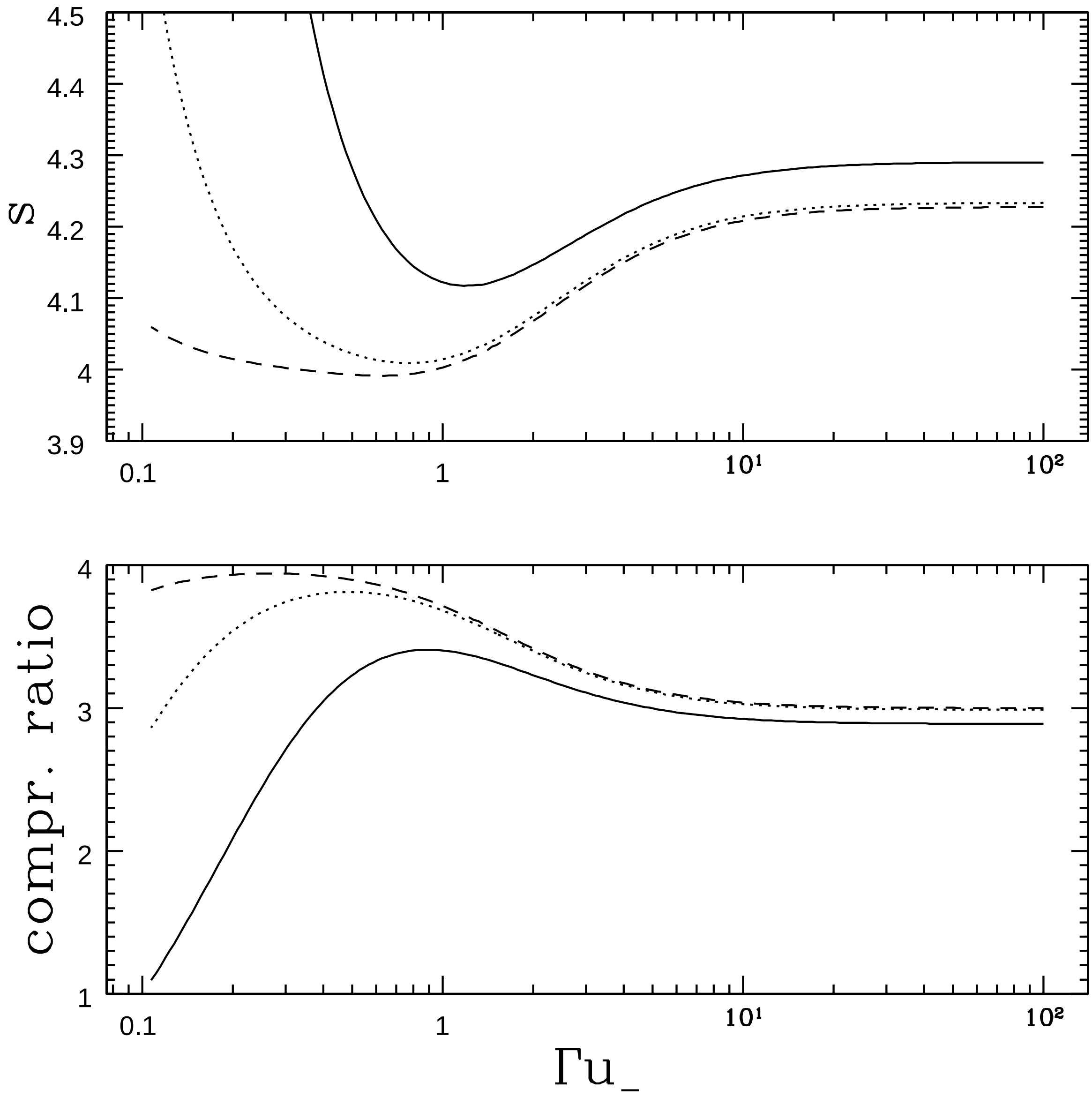

Fig. 4.— The compression ratio and power-law index s of accelerated particles as functions of the spatial component of the upstream four velocity at a strong shock in a magnetized plasma. The curves correspond to $\sigma = 10^{-2}$ (solid line) 10^{-3} (dotted line) and 10^{-4} (dashed line). The Jüttner/Synge equation of state was used and a strong shock assumed.

where v_s is the sound speed in the plasma, in units of the speed of light.

At nonrelativistic shocks, it is usually assumed that Alfvén waves are responsible for scattering and that they propagate away from the shock front, since they are thought to be generated by the streaming of the accelerated particles. This results in an effective reduction in the velocity discontinuity experienced by the particles by an amount which depends on the magnetic field strength, and can have a significant influence on the predicted power-law index. In our case, however, where direction-angle scattering rather than pitch-angle scattering is important, there is no clear connection between the average magnetic field strength and the speed of the scattering centers. Nor is it obvious that the accelerated particles are responsible for generating the fluctuations. In view of this, we make the simple assumption that they are frozen into the background plasma.

Nevertheless, the magnetic field can influence both the angular distribution and the spectrum of accelerated particles by changing the jump conditions across the shock front. These can be computed numerically using a straightforward algorithm (Majorana & Anile 1987; Kirk & Duffy 1999), into which one can incorporate the full Jüttner/Synge equation of state. Figure 4 shows results obtained using this equation of state for cold upstream plasma ($v_s = 0$) for a range of σ values.

The general effect of finite σ is to reduce the compression ratio. This leads to steeper spectra for the accelerated particles, as found by Ballard & Heavens (1991). Note that, in accordance with Eq. (25), the compression in the nonrelativistic limit does not tend to the unmagnetized value, but remains substantially weaker, finally disappearing when the shock speed reaches the fast magnetosonic speed ($M_{\rm fast} = 1$) at $u_- = \sqrt{\sigma/(1+\sigma)}$.

4.3. Anisotropic diffusion in direction angle

The assumption implicit in the above computations is that the magnetic field is tangled on short length scales, such that the particle motion can be described as diffusion in direction angle (the angle between the shock normal and the particle velocity). The diffusion coefficient $D_{\mu\mu} = (1-\mu^2)D$, with D constant, applies in a magnetic field which has no preferred direction in space.

On the other hand, it is reasonable to suppose that the turbulence at a relativistic shock does have a preferred direction – that of the shock normal – leading us to expect anisotropic diffusion as discussed in Sect. 2. Figure 5 shows the angular distribution at the shock front when diffusion proceeds according to Eq. (7) with $\ell_{||}/\ell_\perp = 0.1$. In this figure we also plot the approximate expression for the distribution given in Eq. (23), evaluated using s from the full numerical solution. Although the distribution at $\Gamma_- u_-$ has converged in the sense that it does not change appreciably for higher speeds, it remains significantly different from the approximate expression. This emphasizes the necessity of including higher eigenfunctions in the treatment of highly anisotropic downstream diffusion coefficients. Using the diffusion coefficient in Eq. (7) enables particles at small μ to diffuse rapidly in angle. This leads to a marked change in the form of the distribution which is especially

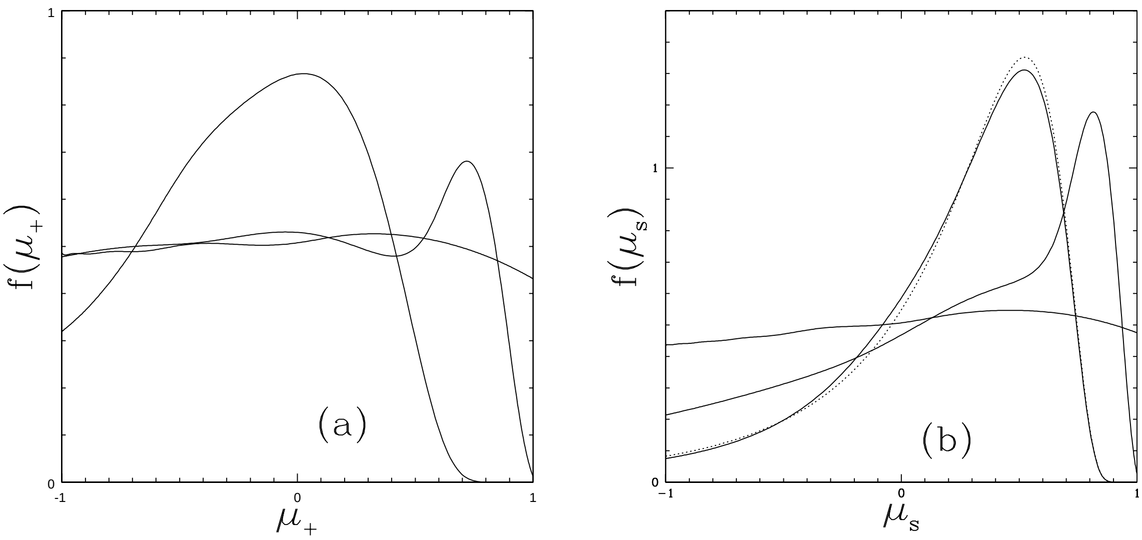

Fig. 5.— The distribution function at the shock front for anisotropic diffusion according to Eq. (7) with $\ell_{||}/\ell_\perp = 0.1$ (a) as seen in the downstream rest frame as a function of the direction angle μ_+ and (b) as seen in the shock rest frame, as a function of the direction angle μ_s. Curves are plotted for three different values of the upstream speed: $\Gamma_- u_- = 0.1$, 1, and 10. The maximum value of f rises monotonically with u_-. For $\Gamma_- u_- \geq 10$, there is no further discernable change in the distribution. In (b) the dotted line shows, for comparison, the approximate expression given in Eq. (23), using $\Gamma_- u_- = 10$, and the appropriate value of s. The jump conditions used are those of a strong shock with the Jüttner/Synge equation of state downstream

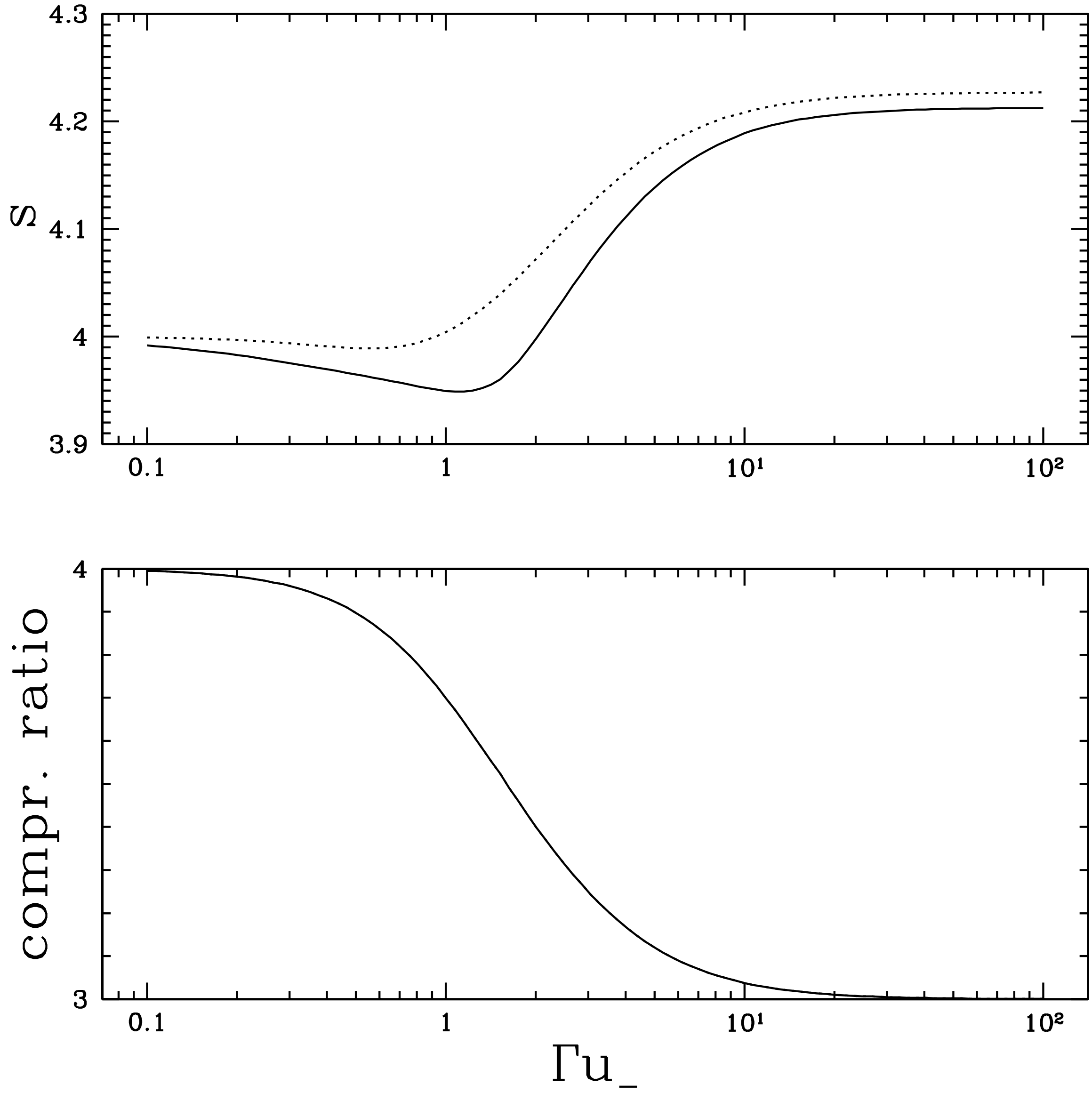

Fig. 6.— The power-law index for anisotropic diffusion at a strong hydrodynamic shock front. The dotted line shows the result for isotropic diffusion, the solid line that for diffusion in a tangled field with short correlation length along the direction of the shock normal, according to Eq. (7) with $\ell_{\|}/\ell_{\perp} = 0.1$. The lower panel shows the compression ratio of the shock. The jumps conditions are the same as those used for Fig. 5

pronounced at intermediate shock speeds. Instead of diffusing preferentially along the shock front, particles at these shocks are rapidly moved to directions closer to the shock normal.

This results in an enhancement of the average momentum gain per crossing and recrossing cycle, but also increases the probability that a particle escapes when on the downstream side of the shock. Results for the power-law index using anisotropic diffusion at a strong shock are presented in Fig. 6 and show harder values than for the isotropic scattering case. The largest effect again appears for mildly relativistic shock speeds, but the difference between isotropic and anisotropic scattering is not dramatic amounting to a decrease in s of at most 0.06. In the ultrarelativistic limit, there is a small but noticeable effect, corresponding to a decrease of about 0.02.

5. Discussion

We present semi-analytic computations of the power-law index of particles undergoing first order Fermi acceleration at ultrarelativistic shocks. In the limit of high shock speed, the index tends to a value of $s = 4.23$ when the magnetic field is dynamically unimportant. This agrees with the results of Monte-Carlo simulations of both highly relativistic shocks (Bednarz & Ostrowski 1998; Achterberg et al 2000) and of the limiting case of ultrarelativistic shocks (Gallant et al 2000). The result is independent of the equation of state of the plasma, since, in the ultrarelativistic limit, the compression ratio of all hydrodynamic shocks tends to 3. If the effects of cooling are negligible, this distribution gives rise to synchrotron radiation with a spectral index $\alpha = 0.62$ (where $\alpha = -\mathrm{d}\ln F_\nu/\mathrm{d}\nu$, F_ν is the flux of radiation and ν the frequency). On the other hand, if the radiative cooling time is short compared to the escape time from the source the observed spectrum steepens to $\alpha = 1.12$. Distributions with spectral indices compatible with this value have been identified as responsible for the synchrotron emission in various objects thought to contain highly relativistic shock fronts: gamma-ray bursts afterglows (Galama et al 1998), the Crab Nebula (Kennel & Coroniti 1984b) and the Blazar Mkn 501 (Krawczynski et al 2000).

Physically, the spectrum is determined by the extent to which particles crossing and re-crossing the shock front form a population whose velocity vectors are beamed predominantly along the shock surface. This 'capturing' by the shock has two effects: on the one hand it reduces the probability that the particles are swept away from the shock front, reducing their escape probability per cycle, but, on the other hand, it also reduces the mean energy gained per cycle (Gallant & Achterberg 1999). For an isotropically distributed incoming population, the relative energy gain upon crossing and re-crossing is approximately Γ_-^2 (Vietri 1995), but for particles which undergo many crossings, this decreases to become of the order of unity.

As in the case of mildly relativistic shocks, anisotropic transport may in principle affect the result. We compute an example in which scattering is due to a tangled magnetic field which is not isotropic, but has a shorter correlation length along the shock normal. The result is a relatively minor correction to the power-law index. The 'universality' of the index 4.23 can, however, be broken if the shock front moves into a strongly magnetized plasma. We present results which indicate that for a value of $\sigma = 10^{-2}$ the asymptotic index for ultrarelativistic shocks steepens to $s = 4.30$. This is mainly a result of the decreased compression ratio of such shocks, a phenomenon which has been described for mildly relativistic shocks by Ballard & Heavens (1991) and may be of importance for acceleration at shocks in pulsar driven winds.

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A. Eigenfunctions

Using the Prüfer transformation:

$$\begin{aligned} Q(\mu) &= \exp[w(\mu)]\sin\Theta(\mu) && \text{(A1)} \\ D(\mu)(1-\mu^2)\frac{\mathrm{d}Q}{\mathrm{d}\mu} &= \exp[w(\mu)]\cos\Theta(\mu) && \text{(A2)} \\ & && \text{(A3)} \end{aligned}$$

the eigenvalue problem Eq. (8) can be transformed into two first order differential equations:

$$\Theta' = \frac{\cos^2\Theta}{D(\mu)(1-\mu^2)} - \Lambda(u+\mu)\sin^2\Theta \tag{A4}$$

$$w' = \sin\Theta\cos\Theta\left[\frac{1}{D(\mu)(1-\mu^2)} + \Lambda(u+\mu)\right] \tag{A5}$$

together with the boundary conditions that Θ and w are regular at $\mu = \pm 1$, i.e.

$$\left.\begin{aligned} \cos\Theta &= 0 \\ \Theta' &= \Lambda(1-u) \\ w' &= -\Lambda(1-u)/2D(-1) \end{aligned}\right\} \text{at } \mu = -1$$

and

$$\left.\begin{aligned} \cos\Theta &= 0 \\ \Theta' &= -\Lambda(1+u) \\ w' &= -\Lambda(1+u)/2D(1) \end{aligned}\right\} \text{at } \mu = +1$$

There are two advantages to this method. Firstly, it is necessary to solve only a single first-order differential equation to find the eigenvalues. Secondly, taking $\Theta = \pi/2$ at $\mu = -1$, each individual eigenvalue Λ_n is found by solving a unique boundary value problem with $\Theta = (2n-1)\pi/2$ at $\mu = +1$. This is easily achieved using a shooting method to match the solutions at the point $\mu = -u$.

For ultrarelativistic flows, one may define the small parameter $\epsilon = 1 - u$ and the ODE's (A4) and (A5) possess two boundary layers. For $\lambda = \epsilon^2\Lambda \sim O(1)$, these have thickness $O(\epsilon)$ at $\mu = -1$

and $O(\epsilon^2)$ at $\mu = +1$. In terms of the stretched variable describing the left-hand layer $y = (1+\mu)/\epsilon$, the zeroth order equation in ϵ reads:

$$\frac{\mathrm{d}\Theta}{\mathrm{d}y} = \frac{\cos^2\Theta}{2yD(-1)} - \lambda \sin^2\Theta(y-1) \tag{A6}$$

with boundary conditions $\cos\Theta = 0$ at $y = 0$ and $\sin\Theta = 0$ at $y \to \infty$. This equation can be transformed into the linear second order differential equation given in Eq. (A4) of Kirk & Schneider (1989), which has the solutions given in Eqs. (14) and (15). However, especially for high-order eigenfunctions, it is more convenient to evaluate the solution by numerical integration of Eqs. (A4) and (A5), taking the approximate boundary condition $\Theta = n\pi$, at $\mu = 1 - O(\epsilon)$.

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